AN INTRODUCTION TO COMPLETENESS OF POSITIVE LINEAR RECURRENCE SEQUENCES

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Abstract. A positive linear recurrence sequence (PLRS) is a sequence defined by a homogeneous linear recurrence relation with positive coefficients and a particular set of initial conditions. A sequence of positive integers is *complete* if every positive integer is a sum of distinct terms of the sequence. One consequence of Zeckendorf's theorem is that the sequence of Fibonacci numbers is complete. Previous work has established a generalized Zeckendorf's theorem for all PLRS's. We consider PLRS's and want to classify them as complete or not. We study how completeness is affected by modifying the recurrence coefficients of a PLRS. Then, we determine in many cases which sequences generated by coefficients of the form $[1, \dots, 1, 0, \dots, 0, N]$ are complete. Further, we conjecture bounds for other maximal last coefficients in complete sequences in other families of PLRS's. Our primary method is applying Brown's criterion, which says that an increasing sequence $\{H_n\}_{n=1}^{\infty}$ is complete if and only if $H_1 = 1$ and $H_{n+1} \leq 1 + \sum_{i=1}^{n} H_i$. This paper is an introduction to the topic that is explored further in [BHLLMT].

Contents

1. Introduction

Edouard Zeckendorf famously proved that every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers, when indexed $\{1, 2, 3, 5, \dots\}$; this unique decomposition is called the *Zeckendorf decomposition* [Ze]. The property of unique decompositions has been generalized to a much larger class of linear recurrence relations, called PLRS's. The following definitions are from [MW, BBGILMT].

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Definition 1.1. *We say a sequence $\{H_n\}_{n=1}^{\infty}$ of positive integers is a* ***Positive Linear Recurrence Sequence (PLRS)*** *if the following properties hold:*

(1) Recurrence relation: *There are non-negative integers $L, c_1, \ldots, c_L$ such that*

$$H_{n+1} \;=\; c_1 H_n + \cdots + c_L H_{n+1-L}, \tag{1.1}$$

with L, c_1 and c_L positive.

(2) Initial conditions: *$H_1 = 1$, and for $1 \le n < L$ we have*

$$H_{n+1} \;=\; c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1. \tag{1.2}$$

Definition 1.2 (Legal decompositions)**.** *We call a decomposition $\sum_{i=1}^{m} a_i H_{m+1-i}$ of a positive integer N (and the sequence $\{a_i\}_{i=1}^{m}$) legal if $a_1 > 0$, the other $a_i \ge 0$, and one of the following two conditions holds:*

(1) *We have $m < L$ and $a_i = c_i$ for $1 \le i \le m$.*

(2) *There exists $s \in \{1, \ldots, L\}$ such that*

$$a_1 \;=\; c_1,\; a_2 \;=\; c_2,\; \cdots,\; a_{s-1} \;=\; c_{s-1} \text{ and } a_s < c_s, \tag{1.3}$$

$a_{s+1}, \ldots, a_{s+\ell} \;=\; 0$ *for some $\ell \ge 0$, and $\{b_i\}_{i=1}^{m-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is legal or empty.*

The following theorem is due to [GT], and is stated in this form in [MW].

Theorem 1.3 (Generalized Zeckendorf's Theorem for PLRS)**.** *Let $\{H_n\}_{n=1}^{\infty}$ be a* Positive Linear Recurrence Sequence*. Then there is a unique legal decomposition for each positive integer $N \ge 0$.*

The goal of this paper is to provide an introduction to the *completeness* of PLRS's. This definition is from [Br, HK].

Definition 1.4. *An arbitrary sequence of positive integers $\{f_i\}_{i=1}^{\infty}$ is* ***complete*** *if and only if every positive integer n can be represented in the form $n = \sum_{i=1}^{\infty} \alpha_i f_i$, where $\alpha_i \in \{0, 1\}$. A sequence that fails to be complete is* ***incomplete****.*

In other words, a sequence of positive integers is complete if and only if each positive integer can be written as a sum of unique terms of the sequence. The Fibonacci numbers are a motivating example.

Example 1.5. *The Fibonacci sequence is complete. This sequence, in particular with the initial conditions given by Definition 1.1, is the PLRS defined by $H_{n+1} = H_n + H_{n-1}$, with $H_1 = 1,\; H_2 = 2$. Here, completeness follows from Zeckendorf's Theorem, as every positive integer has a unique decomposition, and critically, no sequence terms are used more than once. In fact, Zeckendorf's Theorem is a stronger statement than what is required for completeness. Completeness does not require the decompositions to be unique, nor that they use only nonconsecutive terms.*

After seeing this example, does Theorem 1.3 imply that all PLRS's are complete? Previous work in numeration systems by Gewurz and Merola [GM] has shown that specific classes of recurrences as defined by Fraenkel [Fr] are complete under their greedy expression. However, we cannot generalize this result to all PLRS's. For legal decompositions, the decomposition rule might permit sequence terms to be used more than once. This is not allowed for completeness decompositions, where each unique term from the sequence can be used at most once.

Example 1.6. *The PLRS $H_{n+1} = H_n + 3H_{n-1}$ has terms $\{1, 2, 5, 11, \ldots\}$. The unique* legal *decomposition for 9 is $1 \cdot 5 + 2 \cdot 2$, where the term 2 is used twice. However, no* complete

decomposition for 9 *exists. Adding all terms from the sequence less than* 9 *is* $1 + 2 + 5 = 8$*, and to include* 11 *or any subsequent term surpasses* 9*.*

It is not realistic to check that all terms of an infinite sequence have decompositions that use each term no more than once. Instead, we make use of the following criterion for completeness of a sequence, due to [Br]. It allows us to simplify proving completeness for many specific PLRS's to induction proofs.

Theorem 1.7 (Brown's Criterion)**.** *If* a_n *is a nondecreasing sequence, then* a_n *is complete if and only if* $a_1 = 1$ *and for all* $n > 1$*,*

$$a_{n+1} \ \leq \ 1 + \sum_{i=1}^{n} a_i. \tag{1.4}$$

In order to quantify how close a sequence is to the upper bound established by Brown's criterion, we introduce the following definition.

Definition 1.8. *For a sequence* $\{H_n\}$*, we define its* ***nth Brown's gap***

$$B_{H,n} \ := \ 1 + \sum_{i=1}^{n-1} H_i - H_n. \tag{1.5}$$

Notation 1.9. *We use the notation* $[c_1, \ldots, c_L]$*, which is the collection of all* L *coefficients, to represent the PLRS* $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$*.*

A simple case to consider is when all coefficients in $[c_1, \ldots, c_L]$ are strictly positive. The following result, proved in Section 2, completely characterizes these sequences are either complete or incomplete.

Theorem 1.10. *If* $\{H_n\}$ *is a PLRS generated by all positive coefficients* $[c_1, \ldots, c_L]$*, then sequence is complete if and only if the coefficients are* $[\underbrace{1, \ldots, 1}_{L}]$ *or* $[\underbrace{1, \ldots, 1}_{L-1}, 2]$ *for* $L \geq 1$*.*

The situation becomes much more complicated when we consider all PLRS's that have at least one 0 as a coefficient. In order to be able to make progress on determining completeness of these PLRS's, we develop several additional tools. The following three theorems are results that allow certain modifications of the coefficients $[c_1, \ldots, c_L]$ that generate a PLRS that is known to be complete or incomplete, and preserve completeness or incompleteness. They are proved in Section 2.

Theorem 1.11. *Consider sequences* $\{G_n\} = [c_1, \ldots, c_L]$ *and* $\{H_n\} = [c_1, , \ldots, c_L, c_{L+1}]$*, where* c_{L+1} *is any positive integer. If* $\{G_n\}$ *is incomplete, then* $\{H_n\}$ *is incomplete as well.*

Theorem 1.12. *Consider sequences* $\{G_n\} = [c_1, \ldots, c_{L-1}, c_L]$ *and* $\{H_n\} = [c_1, \ldots, c_{L-1}, k_L]$*, where* $1 \leq k_L \leq c_L$*. If* $\{G_n\}$ *is complete, then* $\{H_n\}$ *is also complete.*

Theorem 1.13. *Consider sequences* $\{G_n\} = [c_1, \ldots, c_{L-1}, c_L]$ *and* $\{H_n\} = [c_1, \ldots, c_{L-1} + c_L]$*. If* $\{G_n\}$ *is incomplete, then* $\{H_n\}$ *is also incomplete.*

The next two theorems are results that classify two families of PLRS's as complete or incomplete. They are shown in Section 3.1.

Theorem 1.14. *The sequence generated by* $[1, \underbrace{0, \ldots, 0}_{k}, N]$ *is complete if and only if* $1 \leq N \leq \lceil (k+2)(k+3)/4 \rceil$*, where* $\lceil \cdot \rceil$ *is the ceiling function.*

Theorem 1.15. *The sequence generated by* $[1, 1, \underbrace{0, \ldots, 0}_{k}, N]$ *is complete if and only if* $1 \leq N \leq \lfloor (F_{k+6} - k - 5)/4 \rfloor$*, where* F_n *are the Fibonacci numbers with* $F_1 = 1, F_2 = 2$*, and* $\lfloor \cdot \rfloor$ *is the floor function.*

We have a partial extension of these theorems to when there are g initial ones followed by k zeroes in the collection of coefficients. For a proof, see [BHLLMT].

Theorem 1.16. *Consider a PLRS generated by coefficients* $[\underbrace{1, \ldots, 1}_{g}, \underbrace{0, \ldots, 0}_{k}, N]$*, with* $g, k \geq 1$*.*

(1) *For* $g \geq k + \lceil \log_2 k \rceil$*, the sequence is complete if and only if* $1 \leq N \leq 2^{k+1} - 1$*.*
(2) *For* $k \leq g \leq k + \lceil \log_2 k \rceil$*, the sequence is complete if and only if* $1 \leq N \leq 2^{k+1} - \lceil k/2^{g-k} \rceil$*.*

In Section 3.2, we discuss finitary criteria for completeness, i.e., if there exists a position in the sequence by which Brown's criterion must fail, if it will ever fail. Based on experimental data, we conjecture that such a bound does exist. We do show a weaker result, which is an additional sufficient condition for completeness based on Brown's gap.

Theorem 1.17. *The PLRS* $\{H_n\}$ *generated by* $[c_1, c_2, \ldots, c_L]$ *is complete if*

$$\begin{cases} B_{H,n} \geq 0 & \text{for } n < L \\ B_{H,n} > 0 & \text{for } L \leq n \leq 2L - 1. \end{cases} \tag{1.6}$$

This paper is an introduction to the classification of PLRS's by completeness and serves as an introduction to the full results, which include an analysis of the principal root of the recurrence relation's characteristic function, in [BHLLMT].

2. Modifying Sequences

A basic question to ask is how far we can tweak the coefficients used to generate a sequence, yet preserve its completeness. The modifying process turns out to be well-behaved and heavily dependent on the location of coefficients that are changed. Before we start looking into implementing any changes to our sequences, we first need to understand the maximal complete sequence.

2.1. **The Maximal Complete Sequence.** The maximal complete sequence is the sequence that has terms that grow as quickly as possible while the sequence remains complete. For example, if a sequence begins $\{1, t, \ldots\}$, what can t possibly be for the sequence to be complete? The sequence is increasing as a result of the specific initial conditions we are using, until the full recurrence relation takes over. So except in the degenerate case of $H_{n+1} = H_n$, i.e., the collection of coefficients is just [1], the sequence is strictly increasing. On the other hand, if $t \geq 3$, then there is no way to create a decomposition for 2 that uses sequence terms only once. This means that the maximal complete sequence has $t = 2$. Extending this idea, we establish the following lemma.

Lemma 2.1. *The complete sequence with the maximal possible terms is* $\{a_n\} = \{2^{n-1}\}$*. In other words, any sequence* $\{H_n\}$ *which fulfills* $H_n > 2^{n-1}$ *for some* n *must be incomplete.*

Proof. It is straightforward to see $\{a_n\} = \{2^{n-1}\}$ is generated by the PLRS $H_{n+1} = 2H_n$. This is complete by Brown's criterion, since for any n,

$$2^n = 1 + \sum_{i=1}^{n} 2^{i-1}.$$

Observe that by using a strict equality here with Brown's criterion, we are "maximizing" the complete sequence.

Now, let $\{b_n\}$ be an increasing sequence of positive integers, and suppose for some n, $b_n > 2^{n-1}$, i.e., at some index n, the sequence $\{b_n\}$ exceed that of the sequence $\{2^{n-1}\}$. Note that there are precisely $2^{n-1}-1$ non-empty subsets of $\{b_1, \ldots, b_{n-1}\}$, and thus at most $2^{n-1}-1$ positive integers which can be expressed as a sum of these values. Thus, as the set $\{1, 2, \ldots, b_n - 1\}$ has at least 2^{n-1} elements, at least one of those elements cannot be written as a sum of integers in $\{b_1, \ldots, b_{n-1}\}$, and so the sequence is not complete. Hence, we conclude that $\{2^{n-1}\}$ is the complete sequence such that each term is maximized. □

Now we can look at all complete sequences with only positive coefficients.

Proof of Theorem 1.10. Assuming completeness of the sequence, by the definition of a PLRS and by Brown's criterion, we have

$$c_1H_{L-1} + c_2H_{L-2} + \cdots + c_{L-1}H_1 + 1 \;=\; H_L \leq 1 + H_1 + H_2 + \cdots + H_{L-1}. \tag{2.1}$$

Since $c_i \geq 1$ for $1 \leq i \leq L$, $c_i = 1$ for $1 \leq i < L$. By the definition of a PLRS,

$$H_{L+1} \;=\; c_1H_L + c_2H_{L-1} + \cdots + c_LH_1 \;=\; H_L + H_{L-1} + \cdots + H_2 + c_LH_1. \tag{2.2}$$

Which together with Brown's criterion gives $c_LH_1 \leq 1 + H_1 = 2$. And so $c_L \leq 2$, which completes the forward direction of the proof.

Conversely, we know that the sequence [2] is complete by Lemma 2.1. Thus, let us assume that $c_1 = \cdots = c_{L-1} = 1$ and $1 \leq c_L \leq 2$. We prove that H_n satisfies Brown's criterion. We can show this explicitly for $1 \leq n < L$ and by strong induction on n further on, where the inductive hypothesis is applied to H_{n+1-L} to obtain

$$H_{n+2} \;\leq\; H_{n+1} + \cdots + H_{n+2-L} + H_{n+1-L} + (H_{n-L} + \cdots + H_1 + 1), \tag{2.3}$$

which completes the proof. □

A specific case of Theorem 1.10 is that a PLRS with coefficients $[\underbrace{1, \ldots, 1}_{L-1}, 2]$ is complete. A consequence of Lemma 2.1 it that $\{H_k\} = \{2^{k-1}\}$ is an inclusive upper bound for any complete sequence. A careful reader might note that these two results are related. Due to a PLRS's specific initial conditions, we can prove that this sequence $\{2^{k-1}\}$ can be generated by multiple collections of coefficients. The proof, by strong induction, can be found in [BHLLMT].

Corollary 2.2. *A PLRS with coefficients* $[\underbrace{1, \ldots, 1}_{L-1}, 2]$ *generates the sequence* $H_n = 2^{n-1}$.

2.2. **Modifications of Sequences of Arbitrary Coefficients.** Modifying coefficients in order to preserve completeness turns out to be a balancing act. Sometimes increasing a coefficient causes an incomplete sequence to become complete, while other times, increasing a coefficient causes a complete sequence to become incomplete. For example, $[1, 0, 0, 0, 0, 0, 15]$ is incomplete; increasing the second coefficient to 1, i.e., $[1, 1, 0, 0, 0, 0, 15]$ is complete. Further increasing it to 2, i.e., $[1, 2, 0, 0, 0, 0, 15]$ is again incomplete. To study how such modifications preserve completeness or incompleteness, we take advantage of Brown's gap: a direct consequence of Brown's criterion is that $\{H_n\}$ is complete if and only if $B_{H,n} \geq 0$ for all $n \in \mathbb{N}$.

So, what happens if we append one more coefficient to $[c_1, \ldots, c_L]$? It turns out that if our sequence is already incomplete, appending any new coefficients will never make it complete. This result is Theorem 1.11, which we now prove using Brown's gap.

Proof of Theorem 1.11. By Brown's criterion, it is clear that $\{G_n\}$ is incomplete if and only if there exists n such that $B_{G,n} < 0$. We claim that for all m, $B_{H,m} \leq B_{G,m}$. If true, our lemma is proven: suppose $B_{G,n} < 0$ for some n, we would see $B_{H,n} \leq B_{G,n} < 0$, implying $\{H_n\}$ is incomplete as well.

We proceed by induction. Clearly, $B_{H,k} = B_{G,k}$ for $1 \leq k \leq L$. Further, for $k = L$, we see

$$B_{G,L+1} - B_{H,L+1} \;=\; 1 + \sum_{i=1}^{L} G_i - G_{L+1} - \left(1 + \sum_{i=1}^{L} H_i - H_{L+1}\right) \;=\; H_{L+1} - G_{L+1} = 1 > 0.$$

Now, let $m \geq 2$ be arbitrary, and suppose

$$B_{H,\,L+m-1} \;\leq\; B_{G,\,L+m-1}. \tag{2.4}$$

We wish to show that $B_{H,\,L+m} \leq B_{G,\,L+m}$. Note that

$$B_{H,\,L+m} - B_{H,\,L+m-1} \;=\; 2H_{L+m-1} - H_{L+m}. \tag{2.5}$$

Similarly,

$$B_{G,\,L+m} - B_{G,\,L+m-1} \;=\; 2G_{L+m-1} - G_{L+m}. \tag{2.6}$$

It may be proven through induction that for all $k \geq 2$, $H_{L+k} - G_{L+k} \geq 2\,(H_{L+k-1} - G_{L+k-1})$ (for more details, see Appendix B of [BHLLMT]). Applying it to equations (2.5) and (2.6), we see that $B_{H,\,L+m} - B_{H,\,L+m-1} \leq B_{G,\,L+m} - B_{G,\,L+m-1}$. Summing this inequality to both sides of inequality (2.4), we arrive at $B_{H,L+m} \leq B_{G,L+m}$, as desired. □

Now, we investigate the behavior when we decrease the last coefficient for any complete sequence. In Theorem 1.12, we find that decreasing the last coefficient for any complete sequence preserves completeness.

Proof of Theorem 1.12. Given that $\{G_n\}$ is complete, suppose for the sake of contradiction that there exists an incomplete $\{H_n\}$. Thus, let m be the least such that

$$H_m \;>\; 1 + \sum_{i=1}^{m-1} H_i. \tag{2.7}$$

Simultaneously, as $\{G_n\}$ is complete, by Brown's criterion,

$$G_m \;\leq\; 1 + \sum_{i=1}^{m-1} G_i. \tag{2.8}$$

First, note that for all $n \leq L$, $G_n = H_n$, hence

$$H_m \;=\; G_m \;\leq\; 1 + \sum_{i=1}^{m-1} G_i \;=\; 1 + \sum_{i=1}^{m-1} H_i, \tag{2.9}$$

which contradicts (2.7). Now, suppose $m > L$. But then by substitution of G for H in the first L terms we obtain

$$1 + \sum_{i=1}^{L} H_i \;\geq\; G_m - \sum_{i=L+1}^{m-1} G_i. \tag{2.10}$$

Moreover,

$$H_m \;>\; 1 + \sum_{i=1}^{m-1} H_i \;=\; 1 + \sum_{i=1}^{L} H_i + \sum_{i=L+1}^{m-1} H_i \;\geq\; G_m - \sum_{i=L+1}^{m-1} G_i + \sum_{i=L+1}^{m-1} H_i, \tag{2.11}$$

and thus

$$H_m - \sum_{i=L+1}^{m-1} H_i \;>\; G_m - \sum_{i=L+1}^{m-1} G_i. \tag{2.12}$$

We claim that the opposite of (2.12) is true, arguing by induction on m. For $m = L+1$, we obtain $G_{L+1} \geq H_{L+1}$ as $k_L \leq c_L$. Now, assume that

$$G_m - \sum_{i=L+1}^{m-1} G_i \;\geq\; H_m - \sum_{i=L+1}^{m-1} H_i \tag{2.13}$$

is true for a positive integer m. Using the inductive hypothesis, it then follows that

$$G_{m+1} - \sum_{i=L+1}^{m} G_i \;=\; G_{m+1} - \sum_{i=L+1}^{m-1} G_i - G_m \;\geq\; G_{m+1} - 2G_m + H_m - \sum_{i=L+1}^{m-1} H_i. \tag{2.14}$$

It may be proven through induction that for all $k \in \mathbb{N}$, $H_{L+k+1} - 2H_{L+k} \leq G_{L+k+1} - 2G_{L+k}$. Note

$$G_{m+1} - 2G_m + H_m - \sum_{i=L+1}^{m-1} H_i \;\geq\; H_{m+1} - 2H_m + H_m - \sum_{i=L+1}^{m-1} H_i \;=\; H_{m+1} - \sum_{i=L+1}^{m} H_i, \tag{2.15}$$

which does contradict (2.12) for all $m > L$. Therefore, for all $m \in \mathbb{N}$, we have contradicted (2.7). Hence, $\{H_n\}$ must be complete as well. □

The result above is crucial in our characterization of *families* of complete sequences in Section 3; finding one complete sequence allows us to decrease the last coefficient to find more. Next, we prove two lemmas used in the proof of Theorem 1.13.

Lemma 2.3. *Let $\{G_n\}$ be the sequence defined by $[c_1, \ldots, c_L]$, and let $\{H_n\}$ be the sequence defined by $[c_1, \ldots, c_{L-1} + 1,\ c_L - 1]$. If $\{G_n\}$ is incomplete, then $\{H_n\}$ must be incomplete as well.*

Proof. We claim that for all m, $B_{H,m} \leq B_{G,m}$. This lemma is proven using similar reasoning as for Theorem 1.11. We proceed by induction. Clearly, $B_{H,k} = B_{G,k}$ for $1 \leq k \leq L-1$. Further, for $k = L$, we see

$$B_{G,L} - B_{H,L} \;=\; 1 + \sum_{i=1}^{L-1} G_i - G_L - \left(1 + \sum_{i=1}^{L-1} H_i - H_L\right) \;=\; H_L - G_L = 1 > 0.$$

Now, let $m \geq 0$ be arbitrary, and suppose

$$B_{H,\,L+m} \;\leq\; B_{G,\,L+m}. \tag{2.16}$$

We wish to show that $B_{H,\,L+m+1} \leq B_{G,\,L+m+1}$. Note that

$$B_{H,\,L+m+1} - B_{H,\,L+m} \;=\; 2H_{L+m} - H_{L+m+1}, \tag{2.17}$$

and similarly,

$$B_{G,\,L+m+1} - B_{G,\,L+m} \;=\; 2G_{L+m} - G_{L+m+1}. \tag{2.18}$$

Note that for all $k \geq 0$, $H_{L+k+1} - G_{L+k+1} \geq 2\,(H_{L+k} - G_{L+k})$. Applying it to (2.17) and (2.18), we see $B_{H,\,L+m+1} - B_{H,\,L+m} \leq B_{G,\,L+m+1} - B_{G,\,L+m}$. Summing this inequality to both sides of inequality (2.16), we conclude that $B_{H,L+m+1} \leq B_{G,L+m+1}$, as desired. □

How many times can Lemma 2.3 be applied? The answer is all the way up to $[c_1, \ldots, c_{L-1} + c_L - 1, 1]$, as the last coefficient must remain positive to stay a PLRS.

Lemma 2.4. *Let $\{G_n\}$ be the sequence defined by $[c_1, \ldots, c_{L-1}, 1]$, and let $\{H_n\}$ be the sequence defined by $[c_1, \ldots, c_{L-1} + 1]$. If $\{G_n\}$ is incomplete, then $\{H_n\}$ must be incomplete as well.*

Remark 2.5. *Despite the similarities, Lemma 2.4 is not implied by Lemma 2.3; both are necessary for the proof of Theorem 1.13. Applying Lemma 2.3 $(c_L - 1)$ times proves that if $[c_1, \ldots, c_{L-1}, c_L]$ is incomplete, then $[c_1, \ldots, c_{L-1} + c_L - 1, 1]$ is incomplete; we cannot apply the lemma further while maintaining a positive final coefficient. Hence the case of Lemma* 2.4 *must be dealt with separately, in order to prove Theorem 1.13.*

Proof. The proof is similar to that of Lemma 2.3. We aim to show that $B_{H,m} \leq B_{G,m}$ for all m. Clearly $B_{H,k} = B_{G,k}$ for $1 \leq k \leq L$. Further, for $k = L + 1$, we see

$$B_{G,L+1} - B_{H,L+1} \;=\; \sum_{i=1}^{L} G_i - G_{L+1} - \left(1 + \sum_{i=1}^{L-1} H_L - H_{L+1}\right) \;=\; H_{L+1} - G_{L+1} \;=\; c_1 \;>\; 0. \tag{2.19}$$

Now, let $m \geq 0$ be arbitrary, and suppose

$$B_{H,L+m} \;\leq\; B_{G,L+m}. \tag{2.20}$$

We wish to show that $B_{H,L+m+1} \leq B_{G,L+m+1}$. Note that

$$B_{H,L+m+1} - B_{H,L+m} \;=\; 2H_{L+m} - H_{L+m+1}, \tag{2.21}$$

and similarly

$$B_{G,L+m+1} - B_{G,L+m} \;=\; 2G_{L+m} - G_{L+m+1}. \tag{2.22}$$

It may be proven through induction that for all $k \geq 0$, $H_{L+k+1} - G_{L+k+1} \geq 2\,(H_{L+k} - G_{L+k})$ (for more details, see Appendix B of the full paper). Applying it to equations (2.21) and (2.22), we see $B_{H,L+m+1} - B_{H,L+m} \leq B_{G,L+m+1} - B_{G,L+m}$. Summing this inequality to both sides of inequality (2.20), we conclude that $B_{H,L+m+1} \leq B_{G,L+m+1}$, as desired. □

Using these lemmas, we can now prove Theorem 1.13.

Proof of Theorem 1.13. We apply Lemma 2.3 $c_L - 1$ times, to conclude that if $[c_1, \ldots, c_{L-1}, c_L]$ is incomplete, then $[c_1, \ldots, c_{L-1} + c_L - 1, 1]$ is incomplete. Finally, applying Lemma 2.4, we achieve the desired result. □

3. Families of Sequences

If we recall Theorem 1.12, it says that given a complete PLRS, decreasing the last coefficient preserves its completeness. This raises a natural question: Given the first $L - 1$ coefficients $c_1, c_2, \ldots, c_{L-1}$, what is the maximal N such that $[c_1, c_2, \ldots, c_{L-1}, N]$ is complete? While we are not able to answer this question in all generality, in this section, we begin exploring it.

3.1. **Using 1's and 0's as Initial Coefficients.**

Proof of Theorem 1.14. Suppose that $\{H_n\}$ is complete. By the definition of a PLRS, we can generate the first $k + 2$ terms of the sequence simply: $H_i = i$ for all $1 \leq i \leq k + 2$. For all $n > k + 1$, we can use the recurrence relation

$$H_{n+1} \;=\; H_n + NH_{n-k-1}. \tag{3.1}$$

In the case that $n = k + 3$,

$$H_{k+4} \;=\; H_{k+3} + NH_2 \;=\; H_{k+3} + 2N. \tag{3.2}$$

As $\{H_n\}$ is complete by supposition, by Brown's criterion,

$$H_{k+4} \;\leq\; H_{k+3} + H_{k+2} + \cdots + H_1 + 1. \tag{3.3}$$

By (3.2), we can replace H_{k+4}, so

$$H_{k+3} + 2N \;\leq\; H_{k+3} + H_{k+2} + \cdots + H_1 + 1, \tag{3.4}$$

and isolating N,

$$\begin{aligned} N \;&\leq\; \left[H_{k+2} + H_{k+1} + \cdots + H_1 + 1\right]/2 \\ &=\; \left[(k+2) + (k+1) + \cdots + 1 + 1\right]/2 \\ &=\; \frac{(k+2)(k+3)}{4} + \frac{1}{2} \end{aligned}$$

and as N is an integer,

$$\begin{aligned} &=\; \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor \\ &=\; \left\lceil \frac{(k+2)(k+3)}{4}. \right\rceil \end{aligned} \tag{3.5}$$

Hence, $N \leq \lceil (k+2)(k+3)/4 \rceil$.

We now prove that if $N \leq \lceil (k+2)(k+3)/4 \rceil$, then $\{H_n\}$ is complete. We first show that if $N_{\max} = \lceil (k+2)(k+3)/4 \rceil$, then $\{H_n\}$ is complete. Taking the recurrence relation $H_{n+1} = H_n + N_{\max} H_{n-k-1}$, and applying Brown's criterion gives

$$\begin{aligned} H_{n+1} \;&=\; H_n + N_{\max} H_{n-k-1} \\ &\leq\; H_n + (N_{\max} - 2) H_{n-k-1} + H_{n-k-1} + H_{n-k-2} + \cdots + H_1 + 1. \end{aligned} \tag{3.6}$$

We can prove by induction that $(N_{\max} - 2) H_{n-k-1} \leq H_{n-1} + \cdots + H_{n-k}$, so

$$H_{n+1} \;\leq\; H_n + H_{n-1} + \cdots + H_{n-k} + H_{n-k-1} + H_{n-k-2} + \cdots + H_1 + 1. \tag{3.7}$$

Hence, by Brown's criterion, the sequence $\{H_n\}$ is complete for $N_{\max}$. Lastly, by Theorem 1.12, for all positive $N < \lceil (k+2)(k+3)/4 \rceil$, the sequence is also complete. □

Once we have established a result such as Theorem 1.14, it is often possible to allow small additional adjustments to the coefficients while maintaining completeness. In the following corollary, we show that for $L \geq 6$, if we switch one of the coefficients from 0 to 1 except for the final zero, then the bound on N to maintain completeness is at least as large.

Corollary 3.1. *For $L \geq 6$, given that $[1, 0, \ldots, 0, N]$ is complete, with $N = \lceil L(L+1)/4 \rceil$, then $[1, c_2, \ldots, c_{L-2}, 0, N]$ is complete where $c_i = 1$ for one $i \in \{2, \ldots, L-2\}$, and the rest are 0.*

Proof. We begin with the recurrence relation for fixed a $i \in \{2, \ldots, L-2\}$,

$$H_{n+1} \;=\; H_n + H_{n-i+1} + N H_{n-L+1}. \tag{3.8}$$

Applying Brown's criterion on the term H_{n-L+1} gives

$$H_{n+1} \;\leq\; H_n + H_{n-i+1} + (N-2) H_{n-L+1} + H_{n-L+1} + H_{n-L} + \cdots + H_1 + 1. \tag{3.9}$$

We can prove by induction that $H_{n-i+1} + (N-2) H_{n-L+1} \leq H_{n-1} + \cdots + H_{n-L+2}$, so

$$\leq\; H_n + H_{n-1} + \cdots + H_{n-L+2} + H_{n-L+1} + H_{n-L} + \cdots + H_1 + 1. \tag{3.10}$$

Hence, by Brown's criterion, the sequence is complete for all $L \geq 6$. □

Proof of Theorem 1.15. Suppose that $\{H_n\}$ is complete. Using the definition of a PLRS, the first $k+3$ terms of the sequence can be generated in the same way: $H_i = F_{i+1} - 1$ for all $1 \le i \le k+3$, where F_n is the Fibonacci sequence. Proceeding in a manner similar to the proof of Theorem 1.14, we see that

$$\begin{aligned} H_{k+4} &= H_{k+3} + H_{k+2} + NH_1 = F_{k+5} + N - 2, \\ H_{k+5} &= H_{k+4} + H_{k+3} + NH_2 = F_{k+6} + 3N - 3, \\ H_{k+6} &= H_{k+5} + H_{k+4} + NH_3 = F_{k+7} + 8N - 5. \end{aligned} \tag{3.11}$$

By applying Brown's criterion,

$$\begin{aligned} H_{k+6} &\le H_{k+5} + H_{k+4} + \cdots + H_1 + 1 \\ &= F_{k+6} + 3N - 3 + F_{k+5} + N - 2 + \sum_{i=1}^{k+3} H_i + 1 \\ &= F_{k+7} + 4N - 5 + \sum_{i=1}^{k+3} (F_{i+1} - 1) + 1. \end{aligned} \tag{3.12}$$

Next,

$$F_{k+7} + 8N - 5 \le F_{k+7} + 4N - 5 + \sum_{i=1}^{k+3} (F_{i+1} - 1) + F_1,$$

which implies

$$4N \le \sum_{i=1}^{k+3} (F_{i+1} - 1) + F_1 = \sum_{i=1}^{k+4} F_i + (k+3) = F_{k+6} + (k+5). \tag{3.13}$$

Thus

$$N \le \frac{F_{k+6} - k - 5}{4}, \tag{3.14}$$

and since N is an integer,

$$N \le \left\lfloor \frac{F_{k+6} - k - 5}{4} \right\rfloor. \tag{3.15}$$

Next, we show that if $N = \lfloor (F_{k+6} - k - 5)/4 \rfloor$, then $\{H_n\}$ is complete. The initial conditions can be found easily, and for the later terms we have

$$\begin{aligned} H_{n+1} &= H_n + H_{n-1} + NH_{n-k-2} \\ &\le H_n + (N-2)H_{n-k-2} + H_{n-k-2} + H_{n-k-3} + \cdots + H_1 + 1. \end{aligned}$$

We can show by induction on n that $(N-2)H_{n-k-2} \le H_{n-1} + \cdots + H_{n-k-1}$ for all $n \ge k+3$ and obtain

$$H_{n+1} \le H_n + H_{n-1} + H_{n-2} + \cdots + H_{n-k-1} + H_{n-k-2} + H_{n-k-3} + \cdots + H_1 + 1. \tag{3.16}$$

Hence, by Brown's criterion, this sequence is complete. Lastly, by Theorem 1.12, for all positive $N < \lfloor (F_{k+6} - k - 5)/4 \rfloor$, the sequence is also complete. □

We want to find a more general result for $[\underbrace{1, \ldots, 1}_{g}, \underbrace{0, \ldots, 0}_{k}, N]$, as seen in Figure 1. Interestingly, we see that as we keep k fixed and increase g, the bound increases, and then stays constant from some value of g onward. This observation motivates the following conjecture.

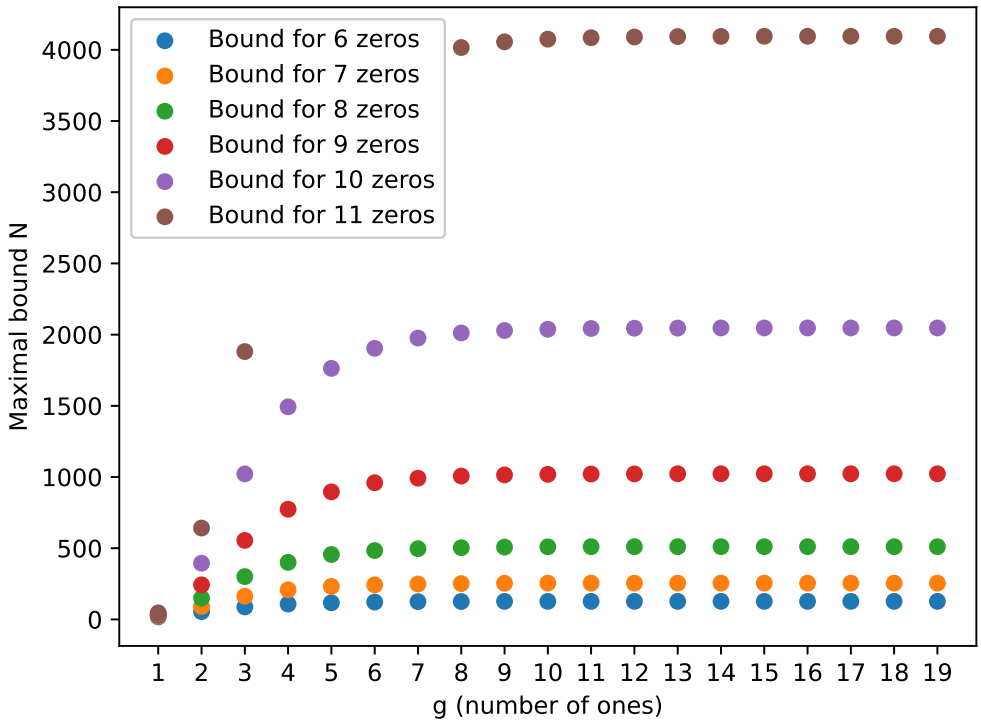

FIGURE 1. $[\underbrace{1,\ldots,1}_{g},\underbrace{0,\ldots,0}_{k},N]$ with k and g varying, where each color represents a fixed k.

Conjecture 3.2. *If* $[\underbrace{1,\ldots,1}_{g},\underbrace{0,\ldots,0}_{k},N]$ *is complete, then so is* $[\underbrace{1,\ldots,1}_{g+1},\underbrace{0,\ldots,0}_{k},N]$.

We have made some progress towards this conjecture; in Theorem 1.16, we showed the precise bound for N when $g \geq k$.

3.2. **Finitary Criteria for Completeness.** Brown's criterion is an efficient way to determine whether a sequence is complete, and as we have seen, many useful results on completeness of PLRS's can be derived from it by induction. However, an arbitrary infinite sequence of positive integers must fulfill Brown's criterion at every term to be considered complete; in order to use it effectively, we wish to bound the number of terms that need to be checked. Thankfully, it can easily be shown that the structure of PLRS's allows us to limit the number of terms for which we need to verify Brown's criterion.

Lemma 3.3. *For any positive integer L, there exists a constant M_L such that any incomplete PLRS generated by L coefficients $[c_1,\ldots,c_L]$ must fail Brown's criterion on or before the M_Lth term.*

Proof. Let $\{H_n\}$ be such a PLRS. Suppose for some index m, we have $c_m > 2^m$, so that $H_{m+1} \geq c_1H_m + \cdots + c_mH_1 > 2^m$. Now, as at least one term of $\{H_n\}$ fulfills $H_n > 2^{n-1}$, we may set m' as the lowest index for which $H_{m'} > 2^{m'-1}$, so that for all $i < m', H_i \leq 2^{i-1}$. Then,

we see that

$$H_{m'} \;>\; 2^{m'-1} \;>\; \sum_{j=1}^{m'-1} 2^{j-1} \;\geq\; \sum_{j=1}^{m'-1} H_j, \tag{3.17}$$

i.e., the sequence fails Brown's criterion at the m'th term.

Thus, if any coefficient c_i exceeds 2^i, the sequence must fail Brown's criterion at one of the first $L+1$ terms. For fixed L, as only finitely many sets of coefficients $[c_1, \ldots, c_L]$ fulfill $c_i \leq 2^i$ for all i, we see that only finitely many incomplete sequences remain which can fail Brown's criterion past the $L+1$st term. Among those, there must be a sequence which fails Brown's criterion last, which gives us the constant M_L. □

In Lemma 3.5 we are able to show that this bound M_L is at least $2L-1$, since $2L-1$ is achieved by $[1, \ldots, 1, 0, 4]$. Moreover, through experimental data, no incomplete sequence has been found to fail for the first time after term $2L-1$, so we conjecture that the bound M_L is exactly $2L-1$.

Conjecture 3.4 (The $2L-1$ Conjecture)**.** *The PLRS $\{H_n\}$ defined by $[c_1, \ldots, c_L]$ is complete if $B_{H,n} \geq 0$ for all $n \leq 2L-1$, i.e., Brown's criterion holds for the first $2L-1$ terms.*

Proving that $[1, \ldots, 1, 0, 4]$ fails at term $2L-1$ and not before is a matter of computing the terms, since we know exactly what the sequence is.

Lemma 3.5. $[1, \ldots, 1, 0, 4]$*, with $k \geq 1$ ones, is always incomplete. Moreover, it first fails Brown's criterion on the $(2k+3)$rd term (equivalently, the $(2L-1)$th term, where L is the number of recurrence coefficients).*

Proof. We can show that $\{H_n\}$ fails Brown's criterion at term $2k+3$ by explicitly computing the first terms of the sequence. The $(2k+3)$rd term is

$$H_{2k+3} \;=\; H_{2k+2} + \cdots + H_{k+3} + 4H_{k+1}; \tag{3.18}$$

for $1 \leq j \leq k+1$, we have $H_j = 2^{j-1}$, and additionally, $H_{k+2} = 2^{k+1} - 1$, so

$$2H_{k+1} \;=\; 2^{k+1} \;>\; 2^{k+1} - 1 \;=\; H_{k+2}; \tag{3.19}$$

and finally, $H_{k+1} = H_k + \cdots + H_1 + 1$. Putting everything together,

$$\begin{aligned} H_{2k+3} \;&=\; H_{2k+2} + \cdots + H_{k+3} + 4H_{k+1} \\ &=\; H_{2k+2} + \cdots + H_{k+3} + 3H_{k+1} + H_k + \cdots + H_1 + 1 \\ &>\; H_{2k+2} + \cdots + H_{k+3} + H_{k+2} + H_{k+1} + H_k + \cdots + H_1 + 1. \end{aligned} \tag{3.20}$$

Hence, $[1, \ldots, 1, 0, 4]$ is incomplete and in particular, Brown's criterion is failed by the $(2k+3)$rd term.

Conversely, through a similar computation, we can show Brown's criterion holds for the first $2k+2$ terms. For $1 \leq j \leq k+1$, we have $H_j = 2^{j-1}$, which are the first terms of the complete sequence $\{2^n\}$. On the other hand, when $k+2 \leq j \leq 2k+2$, we have $1 \leq j-k-1 \leq k+1$, so

$$\begin{aligned} H_{j+1} \;&=\; H_j + \cdots + H_{j-k+1} + 4H_{j-k-1} \\ &=\; H_j + \cdots + H_{j-k+1} + 2H_{j-k-1} + H_{j-k-1} + (H_{j-k-2} + \cdots + H_1 + 1) \\ &=\; H_j + \cdots + H_{j-k+1} + H_{j-k} + H_{j-k-1} + H_{j-k-2} + \cdots + H_1 + 1 \end{aligned} \tag{3.21}$$

as $2H_{j-k-1} = 2^{j-k+1} = H_{j-k}$. So Brown's criterion fails for the first time at term $2k+3$. □

We could also reframe this discussion as a question of when Brown's gap $B_{H,n}$ falls below 0 for the first time. Our conjecture is then that if $\{H_n\}$ is an incomplete PLRS generated by $[c_1, \dots, c_L]$, then $B_{H,n} < 0$ for some $n < 2L-1$.

While this remains a conjecture, we show a weaker result, Theorem 1.17, which has the additional requirement that $B_{H,n} \neq 0$ for all $L \leq n \leq 2L-1$. That is to say, checking that $B_{H,n} \geq 0$ for $n < L$ and $B_{H,n} > 0$ for $L \leq n \leq 2L-1$ is a sufficient criterion for completeness.

Proof of Theorem 1.17. For $L = 1$, an incomplete sequence $[c]$ fails at the second term if and only if $c > 2$. So, we may assume $L \geq 2$. If $c_1 \geq 2$, then the sequence is incomplete as $H_2 = 3$ and 2 has no representation as a sum of term H_i. So we may assume $c_1 = 1$. We show by induction on n that $B_{H,n} > 0$ when $n \geq L$. Suppose $B_{H,n} > 0$ for $L \leq n \leq m$ (with $m \geq 2L-1$). Then

$$\begin{aligned}
B_{H,m+1} &= 1 + \sum_{i=1}^{m} H_i - H_{m+1} \\
&= 1 + \sum_{i=1}^{L} H_i + \sum_{i=L+1}^{m} \left(H_{i-1} + \sum_{j=2}^{L} c_j H_{i-j} \right) - \left(H_m + \sum_{j=2}^{L} c_j H_{m+1-j} \right) \\
&= \left(1 + \sum_{i=1}^{m-1} H_i - H_m + H_L \right) + \sum_{j=2}^{L} c_j \left(\sum_{i=L+1}^{m} H_{i-j} - H_{m+1-j} \right) \\
&= (B_{H,m} + H_L) + \sum_{j=2}^{L} c_j \left(B_{H,m+1-j} - 1 - \sum_{i=j+1}^{L} H_{i-j} \right) \\
&= B_{H,m} + \sum_{j=2}^{L} c_j (B_{H,m+1-j} - 1) + H_L - \sum_{i=3}^{L} \sum_{j=2}^{i-1} c_j H_{i-j} \\
&= B_{H,m} + \sum_{j=2}^{L} c_j (B_{H,m+1-j} - 1) + H_L - \sum_{i=3}^{L} (H_i - H_{i-1} - 1) \\
&= B_{H,m} + \sum_{j=2}^{L} c_j (B_{H,m+1-j} - 1) + L. \qquad (3.22)
\end{aligned}$$

The last line is positive since $B_{H,m+1-j} - 1 \geq 0$ and $B_{H,m}, L > 0$. This completes the induction; hence $\{H_n\}$ is complete. □

The benefit of Conjecture 3.4 is that it would be a necessary and sufficient condition, while Theorem 1.17 provides a weaker sufficient condition. The condition that a PLRS not fail Brown's criterion in the first $2L-1$ terms is certainly necessary for it to be complete; the conjecture is that this would also be sufficient for the sequence to be complete.

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