Analytic Approaches to Completeness of Generalized Fibonacci Sequences

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Introduction

Positive Linear Recurrence Sequences

Definition

A sequence $\{H_i\}_{i\geq 1}$ of positive integers is a **Positive Linear Recurrence Sequence (PLRS)** if the following properties hold:

- (Recurrence relation) There are non-negative integers $L, c_1, \ldots, c_L$ such that

$$H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$$

  with L, c_1, c_L positive.
- (Initial conditions) $H_1 = 1$, and for $1 \leq n < L$,

$$H_{n+1} = c_1 H_n + \cdots + c_n H_1 + 1$$

Positive Linear Recurrence Sequences

- We write $[c_1, \ldots, c_L]$ for $H_{n+1} = c_1 H_n + \cdots + c_L H_{n-L+1}$.
- For example, for the Fibonacci numbers, we write $[1, 1]$. This definition gives initial conditions $F_1 = 1,\ F_2 = 2$.
- Despite satisfying positive linear recurrences, the Lucas and Pell numbers are not PLRS, since their initial conditions do not meet the definition.

Definition

A sequence $\{H_i\}_{i\geq 1}$ is called **complete** if every positive integer is a sum of its terms, using each term at most once.

- The sequence with the recurrence $[1, 3]$ is *not* complete. Its terms are $\{1, 2, 5, 11, \dots\}$; you cannot get 4 or 9 as the sequence grows too quickly.
- The Fibonacci sequence $[1, 1]$, with initial conditions $F_1 = 1$, $F_2 = 2$, is complete (follows from Zeckendorf's Theorem).

The Doubling Sequence

The PLRS [2], which has the recurrence $H_{n+1} = 2H_n$, has terms $H_n = 2^{n-1}$ and is complete because every integer has a binary representation.

Theorem (Brown)

The complete sequence with maximal terms is $H_n = 2^{n-1}$.

Proof.

Suppose $\{G_n\}$ has $G_k > 2^{k-1}$ for some k. As there are only $2^{k-1} - 1$ different ways to sum the terms $G_1, \ldots, G_{k-1}$, some integer in the set $\{1, \ldots, G_k - 1\}$ cannot be written as a sum of terms of $\{G_n\}$. $\square$

Brown's Criterion

Theorem (Brown)

A nondecreasing sequence $\{H_i\}_{i\geq 1}$ is complete if and only if $H_1 = 1$ and for every $n \geq 1$,

$$H_{n+1} \leq 1 + \sum_{i=1}^{n} H_i.$$

- $[1, 0, 1, 0, 0, 12]$ has terms $\{1, 2, 3, 5, 8, 12, 29, 61, \ldots\}$ and so computing the sums $\sum_{i=1}^{n} H_i + 1$ we see $\{2, 4, 7, 12, 20, 32, 61, \ldots\}$
- $[1, 1, 1, 0, 0, 12]$ has terms $\{1, 2, 4, 8, 15, 28, 63, \ldots\}$ and so computing the sums $\sum_{i=1}^{n} H_i + 1$ we see $\{2, 4, 8, 16, 31, 59, \ldots\}$

Binet's Formula and Generalizations

Characteristic Polynomials

Definition

For a PLRS $\{H_n\}$ defined by $[c_1, \ldots, c_L]$, define the characteristic polynomial

$$p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}.$$

- By Descartes's Rule of Signs, $p(x)$ must have precisely one positive root, which we call its **principal root**.
- The principal root of the PLRS is always the largest, i.e., for any root $z \in \mathbb{C}$, $|z| < r$.

Binet's Formula

Theorem (Binet)

The terms $F_1, F_2, \ldots$ of the Fibonacci sequence can be calculated explicitly as

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - (1 - \varphi)^n \right),$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ denotes the Golden Ratio.

- Note that $\varphi,\ 1 - \varphi$ are the roots of the characteristic polynomial $p(x) = x^2 - x - 1$ of this sequence.

Can we get a similar result for a generic PLRS?

Generalized Binet's Formula

Theorem (Generalized Binet's Formula)

If $r_1, \ldots, r_k$ are the distinct roots of the characteristic polynomial of a linear recurrence $\{H_n\}$, with multiplicities $m_1, \ldots, m_k$, then there exist polynomials $q_1, \ldots, q_k$ with $\deg(q_i) \leq m_i - 1$ for which

$$H_n = q_1(n)r_1^n + \ldots + q_k(n)r_k^n.$$

- If $\{H_n\}$ is a PLRS, we can let r_1 be its principal root; since $m_1 = 1$ and for all $i, r_1 > |r_i|$, we have that $H_n = \mathcal{O}(r_1^n)$.

Slow- and Fast-Growing Sequences

- From Generalized Binet's Formula, we know $H_n = \mathcal{O}(r_1^n)$, so the asymptotic growth of $\{H_n\}$ is determined by r_1.

- Generally speaking, complete sequences must grow relatively slowly. Can we relate the size of r_1 to completeness?

Bounding the Principal Root

First Bounds on r_1

Recall the definition $p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}$.

As the constant term c_L is a positive integer, we know $r_1 > 1$; otherwise, as $c_L = \prod r_i^{m_i}$, and for all $i \geq 2$, $|r_i| < r_1$, we would have $0 < |c_L| < 1$.

Lemma (SMALL 2020)

If H_n is a complete PLRS and r_1 is its principal root, then $r_1 \leq 2$.

Proof.

Otherwise, as $H_n = \mathcal{O}\left(r_1^n\right)$, for large n our terms would exceed the maximal sequence $\{2^{n-1}\}$. $\square$

Note: $r_1 \leq 2$ is necessary, but not sufficient!

Is 2 a Useful Bound?

Is 2 the best upper bound for roots of complete sequences?

- 2 is optimal: we can find complete sequences with roots of sizes arbitrarily close to 2, and even with roots of size exactly 2. (Sequences of the form $[\underbrace{1, \ldots, 1}_{m}]$.)
- Checking $r_1 \leq 2$ is a fast method to eliminate candidates for completeness. How to do this effectively?
- As $p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}$ has exactly one positive root, and $p(x) > 0$ for large x, we see $r_1 \leq 2$ if and only if $p(2) \geq 0$. This is much faster than checking Brown's Criterion!

Lower Bound

Lemma (SMALL 2020)

For any L, there exists a second bound B_L such that if a sequence $[c_1, \dots, c_L]$ is incomplete, then $r_1 \geq B_L$.

Proof.

- There are finitely many sequences $[c_1, \dots, c_L]$ with $p(2) = 2^L - \sum_{i=1}^{L} c_i 2^{L-i} \geq 0$. For example, if any $c_i > 2^i$, we have $p(2) < 0$.
- There are finitely many incomplete sequences with $r_1 \leq 2$, and so we can always find the incomplete sequence with smallest root - this is B_L.

$\square$

We now aim to determine the precise values of B_L.

A Few Combinatorial Results

Theorem (SMALL 2020)

If $[c_1, \ldots, c_L]$ is any incomplete sequence, then the sequence $[c_1, \ldots, c_{L-1} + c_L]$ is also incomplete.

Theorem (SMALL 2020)

If a sequence $[c_1, \ldots, c_{L-1}, c_L]$ is complete, then so is $[c_1, \ldots, c_{L-1}, d_L]$ for any $1 \leq d_L \leq c_L$.

Remark. This is not true for c_i in an arbitrary position.

Both can be proven by working directly with Brown's Criterion.

The Minimal Incomplete Sequence

Theorem (SMALL 2020)

$[1, \underbrace{0, \ldots, 0}_{L-2}, N]$, *is complete if and only if*

$$N \leq \left\lceil \frac{L(L+1)}{4} \right\rceil.$$

Conjecture (SMALL 2020)

For any given L, the incomplete sequence of length L with the lowest principal root is $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.

- We denote by λ_L the principal root of $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$. The conjecture is equivalent to saying $\lambda_L = B_L$, for all L.

Even in the event the conjecture is false, asymptotic work on the λ_L gives us useful information for the bound B_L.

Theorem (SMALL 2020)

For $L \in \mathbb{Z}_+$, let λ_L be the sole positive root of

$$p_L(x) = x^L - x^{L-1} - \left\lceil \frac{L(L+1)}{4} \right\rceil .$$

Then, for any L, $\lambda_L > \lambda_{L+1}$. Additionally, $\lim_{L\to\infty} \lambda_L = 1$.

Both of these results can be computed algebraically.

This shows $\lim_{L\to\infty} B_L = 1$, so we can get incomplete sequences that grow arbitrarily slowly. If our conjecture holds, then we get the specific asymptotic behavior L, $B_L \approx (L/2)^{2/L}$.

Proving the Conjecture

We first show any sequence $[c_1, \ldots, c_L]$ where $\sum c_i$ is sufficiently large must have root greater than λ_L.

Case 1: $\sum_{k=1}^{L} c_k \geq 2 + \left\lceil \frac{L(L+1)}{4} \right\rceil$

We combine the following two invariant arguments:

- The principal root of $[c_1, \ldots, c_L]$ is strictly greater than that of $[c_1, \ldots, c_k - 1, \ldots, c_L + 1]$, for any k.
- The principal root of $[1, 0, \ldots, 0, S]$ is strictly greater than that of $[1, 0, \ldots, 0, S - 1]$.

Combining these two, any sequence with large sum can be "reduced" to $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.

Inducting for the General Case

Conjecture

If $[c_1, \ldots, c_L]$ *is an incomplete sequence with* $\sum_{i=1}^{L} c_i \leq \left\lceil \frac{L(L+1)}{4} \right\rceil + 2$, *then its principal root is at least* λ_L.

Base Case: For $L = 2$, we see $\lceil L(L+1)/4 \rceil + 1 = 3$, and so we consider $[c_1, c_2]$ with $c_1 + c_2 \leq 4$. The only incomplete sequences here are $[2, 1], [2, 2], [1, 3], [3, 1]$, with roots 2.414, 2.731, 2.303, 3.303. The smallest corresponds to $[1, 3] = [1, \lceil (2 \cdot 3)/4 \rceil + 1]$, and so the Lemma holds.

Inducting for the General Case

We use strong induction. Suppose the lemma holds for all lengths up to L, and let $[c_1, \ldots, c_L, c_{L+1}]$ be an incomplete sequence with $\sum_{i=1}^{L+1} c_i \leq \left\lceil \frac{(L+1)(L+2)}{4} \right\rceil + 2$.

- We can show analytically that the root of $[c_1, \ldots, c_L, c_{L+1}]$ is greater than that of $[c_1, \ldots, c_L]$. Thus if $[c_1, \ldots, c_L]$ is incomplete, its root exceeds λ_L by induction hypothesis, and so the root of $[c_1, \ldots, c_L, c_{L+1}]$ exceeds λ_{L+1}.
- If $\sum_{i=1}^{L} c_i > \lceil L(L+1)/4 \rceil + 2$, a similar argument shows the root of $[c_1, \ldots, c_L, c_{L+1}]$ exceeds λ_{L+1}.

Thus we are reduced to the case where $[c_1, \ldots, c_L]$ is complete and has $\sum_{i=1}^{L} c_i \leq \lceil L(L+1)/4 \rceil + 2$.

Remaining Case

We we have reduced this to the case where $[c_1, \ldots, c_L]$ is complete and has $\sum_{i=1}^{L} c_i \leq \lceil L(L+1)/4 \rceil + 2$, yet $[c_1, \ldots, c_L, c_{L+1}]$ is incomplete. As $[c_1, \ldots, c_k]$ has root below λ_k for all k, we at least have that for any $1 \leq k \leq L+1$,

$$\sum_{i=2}^{k} c_i \leq \left\lceil \frac{k(k+1)}{4} \right\rceil + 1.$$

If $[c_1, \ldots, c_L, c_{L+1}]$ is incomplete, then by previous result, $[c_1, \ldots, c_L + c_{L+1}]$ is incomplete too. Thus root of $[c_1, \ldots, c_L + c_{L+1}]$ exceeds λ_L, yet root of $[c_1, \ldots, c_L, c_{L+1}]$ is below λ_{L+1}, from which we get

$$\sum_{i=2}^{L} c_i \left(\lambda_{L+1}^{L+1-i} - \lambda_L^{L-i}\right) < \frac{L+2}{2}.$$

Remaining Case

Using the bound

$$\sum_{i=2}^{L} c_i \left(\lambda_{L+1}^{L+1-i} - \lambda_L^{L-i}\right) < \frac{L+2}{2},$$

we see through asymptotic work that this forces the first 32.5% of the c_i to be 0 (excluding c_1).

All experimental evidence for values of L up to 30 suggest that under these conditions, $[c_1, \ldots, c_L, c_{L+1}]$ is only incomplete for huge values of c_{L+1}: much too big for the bounds on $\sum c_i$ to hold.

$$[1, \underbrace{0, \ldots, 0}_{19}, 116] \quad [1, \underbrace{0, \ldots, 0}_{9}, 32, \underbrace{0, \ldots, 0}_{9}, 2932].$$

Denseness of Principal Roots in $[1, 2]$

Denseness of Incomplete Roots

Theorem (SMALL 2020)

For any $L \in \mathbb{Z}^+$, let R_L be the set of roots of all incomplete PLRS of length L. Then, for any $\varepsilon > 0$, there exists an M such that for all $L > M$, for any ε-ball $B_\varepsilon \subset [1, 2]$, $B_\varepsilon \cap R_L \neq \varnothing$.

Corollary

The set $R = \bigcup_{L=1}^{\infty} R_L$ of all principal roots of incomplete sequences is dense in $[1, 2]$.

Proof of Denseness Theorem

We use the fact that the λ_L roots are decreasing and fulfill $\lim_{L\to\infty} \lambda_L = 1$.

Proof.

We analyze the set of the roots of the following list of incomplete sequences:

$$[1, 0, \ldots, 0, \left\lceil \tfrac{L(L+1)}{2} \right\rceil + 1],\ [1, 0, \ldots, 0, \left\lceil \tfrac{L(L+1)}{2} \right\rceil + 2],\ \ldots, [1, 0, \ldots, 0, 2^L]$$

We know the root of the first sequence approaches 1. We can show that the roots of consecutive sequence increase at a decreasing rate. Thus for $\lambda_L < 1 + \varepsilon$, we see roots are going up by at most ε. Since the root of the last sequence exceeds 2, the roots will go through every ε-ball in $(1, 2)$. $\square$

Denseness of Complete Roots

Conjecture (SMALL 2020)

Let C be the set of roots of complete PLRS. Then, C is dense in the interval $(1, 2)$.

- Although we have not been able to prove this rigorously, it seems that a similar argument as before is possible, only considering a different set of sequences, namely those of the form

$$[1, 0, \ldots, 0, \underbrace{1, \ldots, 1}_{m}, N].$$

Conclusion

Conclusion

Here, we have developed:

- A much more computationally efficient way to check completeness for most sequences. Bounding root size is $\mathcal{O}\left(L^2\right)$ as it amounts to evaluating polynomial, checking Brown's Criterion is a $\mathcal{O}\left(2^L\right)$ problem.
- A narrowing-down to the precise interval where complete and incomplete sequences interact.
- Proof that complete and incomplete sequences are evenly spread out throughout that interval.

Future Work: Proving the remaining conjectures in the presentation.

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Proof of Brown's Criterion

Theorem (Brown)

If a_n is a nondecreasing sequence, then a_n is complete if and only if $a_1 = 1$ and for all $n > 1$,

$$a_{n+1} \leq 1 + \sum_{i=1}^{n} a_i.$$

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers, not necessarily distinct, such that $a_1 = 1$ and

$$a_{n+1} \leq 1 + \sum_{i=1}^{n} a_i$$

for $n \in \{1, 2, \ldots\}$. Then for $0 < n < 1 + \sum_{i=1}^{k} a_i$ there exists $\{b_i\}_{i=1}^{k}$, $b_i \in \{0, 1\}$ such that $n = \sum_{i=1}^{k} b_i a_i$.

Proof of Brown's Criterion

We proceed by induction on k. The claim clearly holds for $k = 1$, so assume that it holds for some $k = N$. Hence, we must show that $0 < n < 1 + \sum_{i=1}^{N+1} a_i$ implies the existence of $\{\gamma_i\}_{i=1}^{N+1}$, $\gamma_i \in \{0, 1\}$ such that $n = \sum_{i=1}^{N+1} \gamma_i a_i$. Due to the inductive hypothesis, we only consider values satisfying

$$1 + \sum_{i=1}^{N} a_i \leq n < 1 + \sum_{i=1}^{N+1} a_i.$$

Note that

$$n - a_{N+1} \geq 1 + \sum_{i=1}^{N} a_i - a_{N+1} \geq 0$$

by assumption. Now, if $n - a_{N+1} = 0$, the conclusion follows.

Proof of Brown's Criterion

Otherwise,

$$0 < n - a_{N+1} < 1 + \sum_{i=1}^{N} a_i$$

implies the existence of $\{b_i\}_{i=1}^{N}$ such that $n - a_{N+1} = \sum_{i=1}^{N} b_i a_i$. Then the result is immediate on transposing a_{N+1} and identifying $\gamma_i = b_i$ for $i \in \{1, \ldots, N\}$ and $\gamma_{N+1} = 1$. This completes the sufficiency part of the proof.

Proof of Brown's Criterion

For the necessity, assume that there exists $n_0 \geq 1$ such that $a_{n_0+1} \geq 1 + \sum_{i=1}^{n_0} a_i$. Then, however,

$$a_{n_0+1} > a_{n_0+1} - 1 > \sum_{i=1}^{n_0} a_i,$$

which implies that the positive integer $a_{n_0+1} - 1$ cannot be represented in the form $\sum_{i=1}^{k} b_i a_i$. This leads to a contradiction and completes the proof.